Connection Probabilities for Random-Cluster Model

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Based on a joint work with

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- Eveliina Peltola (Aalto University, University of Bonn)

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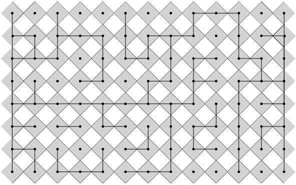
Random-cluster model

Random-cluster model [Fortuin-Kasteleyn, 1970s]

- $G = (V, E)$ is a finite graph.
- $\omega = (\omega_e)_{e \in E} \in \{0,1\}^E$:
  $e \in E$ is open (resp. close) if $\omega_e = 1$ (resp. if $\omega_e = 0$).
- Edge-weight : $p \in (0,1)$. Cluster-weight : $q > 0$.

$$\mathbb{P}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}.$$



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- $q = 2$: FK-Ising model

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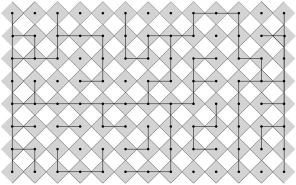

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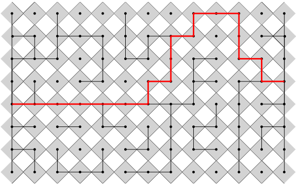

Critical edge-weight [Beffara-Duminil-Copin, PTRF2012]

$$\text{For } q \geq 1, \quad p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

- sub-critical $p < p_c(q)$: connection probability $\to 0$.
- critical $p = p_c(q)$: connection probability is non-trivial.
- super-critical $p > p_c(q)$: connection probability $\to 1$.

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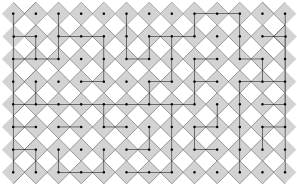

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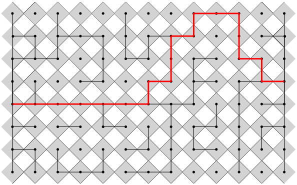

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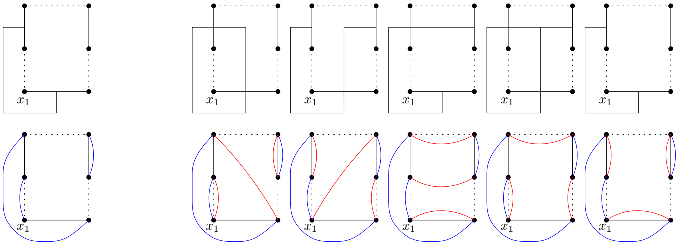
- $q = 1$: Cardy's formula
  Proved by Smirnov for site percolation on $\mathbb{T}$
- $q = 2$: [Chelkak-Smirnov, Invent.2012]

Connection Probabilities



- A polygon $(\Omega; x_1, \ldots, x_{2N})$
- Alternating boundary conditions : $(x_1 x_2), (x_3 x_4), \ldots, (x_{2N-1} x_{2N})$ are wired.
- N wired arcs are further wired according to a non-crossing partition outside of the polygon.

Such boundary condition can be encoded by a planar link pattern with N links of $\{1, 2, \ldots, 2N\}$

$$\text{Boundary condition} : \beta = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_N, b_N\}\}.$$

- N interfaces inside of the polygon : planar link pattern with N links of $\{1, 2, \ldots, 2N\}$

$$\text{Internal link pattern} : \alpha = \{\{c_1, d_1\}, \{c_2, d_2\}, \ldots, \{c_N, d_N\}\}.$$

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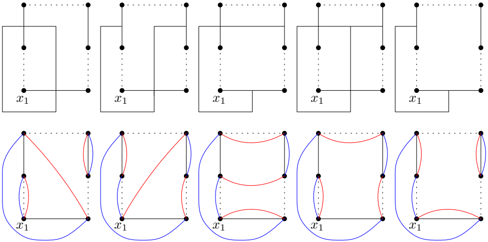


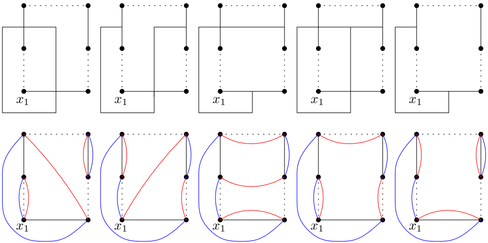
Conjecture on critical random-cluster model [Flores-Kleban-Simmons-Ziff, 2011, 2017]

Fix parameters $\kappa \in (4, 8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0, 4)$. We have the scaling limit of the connection probabilities :

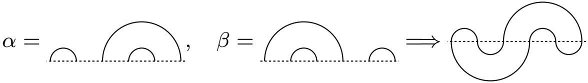
$$\lim_{\delta \to 0} \mathbb{P}^{\delta}_{\beta}[\mathcal{A}^{\delta} = \alpha] = \mathcal{M}_{\alpha,\beta}(q) \frac{\mathcal{Z}_{\alpha}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{G}_{\beta}(\Omega; x_1, \ldots, x_{2N})}.$$

- $\mathcal{M}_{\alpha,\beta}(q)$: Meander matrix
- $\mathcal{G}_{\beta}$: Coulomb gas integrals
- $\mathcal{Z}_{\alpha}$: Pure partition functions

Meander matrix



- Meander formed from α, β :



- Meander matrix

$$\mathcal{M}_{\alpha,\beta}(q) = \sqrt{q}^{\#\text{loops in the meander } (\alpha,\beta)}, \quad \alpha, \beta \in \mathrm{LP}_N.$$

Coulomb gas integrals

- Write the boundary condition as
  $\beta = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_N, b_N\}\}$ with $a_1 < a_2 < \cdots < a_N$ and $a_r < b_r$ for all $1 \leq r \leq N$.

Coulomb gas integrals with $\kappa \in (4, 8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0, 4)$

$$\mathcal{G}_\beta : \ \{\boldsymbol{x} := (x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} : x_1 < \cdots < x_{2N}\} \to \mathbb{C},$$

$$\mathcal{G}_\beta(\boldsymbol{x}) := \left(\frac{\sqrt{q}\,\Gamma(2 - 8/\kappa)}{\Gamma(1 - 4/\kappa)^2} \right)^N \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{2/\kappa}$$
$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1 \leq r < s \leq N} (u_s - u_r)^{8/\kappa} \prod_{\substack{1 \leq i \leq 2N \\ 1 \leq r \leq N}} (u_r - x_i)^{-4/\kappa},$$

where the branch of the multivalued integrand is chosen to be real and positive when

$$x_{a_r} < \mathrm{Re}(u_r) < x_{a_r + 1} \qquad \text{for all } 1 \leq r \leq N.$$

- **BPZ equations** :

$$\left[\frac{\kappa}{2} \partial_i^2 + \sum_{j \neq i} \left(\frac{2}{x_j - x_i} \partial_j - \frac{(6 - \kappa)/\kappa}{(x_j - x_i)^2} \right) \right] F(x_1, \ldots, x_{2N}) = 0.$$

Pure Partition Functions

Pure Partition Functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0$, where $h = (6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1,\ldots,x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1),\ldots,\varphi(x_{2N}))$.

ASY : $\lim_{x_j,x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}), & \text{if } \{j,j+1\}\in\alpha;\\ 0, & \text{else.}\end{cases}$

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- PDE : Itô's formula
- ASY : compatible

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- PDE : 2N variables, 2N PDEs
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Questions

Existence and uniqueness ?

Pure partition functions

Uniqueness [Flores-Kleban, CMP2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP2016]
- $\kappa \in (0, 4]$ [Peltola-W. CMP2019, Beffara-Peltola-W. AOP2021]
- $\kappa \in (0, 6]$ [W. CMP2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

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Theorem [W. CMP2020]

Fix $\kappa \in (0,6]$. The pure partition functions are the recursive collection $\{\mathcal{Z}_\alpha : \alpha \in \cup_N \mathrm{LP}_N\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}$ uniquely determined by the following properties :

PDE, COV, ASY as well as **PLB** :

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{\{a,b\}\in\alpha} |x_b - x_a|^{-2h}, \quad \forall (x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}.$$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly independent and forms a basis for the solution space.

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$

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Fix parameters $\kappa \in (4,6]$ and $q = 4\cos^2(4\pi/\kappa) \in [1,4)$. We have

$$\mathcal{G}_\beta(\boldsymbol{x}) = \sum_{\alpha} \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_\alpha(\boldsymbol{x}).$$

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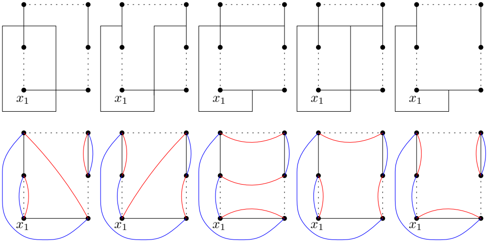
- $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly indept.
- $\{\mathcal{M}_{\alpha,\beta}(q) : \alpha,\beta \in \mathrm{LP}_N\}$ may be not invertible.
- When $\kappa = 16/3$ and $q = 2$, it is NOT invertible.

Connection Probabilities



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Theorem [Feng-Peltola-W. 2022]

Fix $\kappa = 16/3$ and $q = 2$. We have the scaling limit of the connection probabilities :

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Moreover, when $\Omega = \mathbb{H}$,

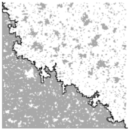
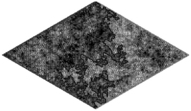
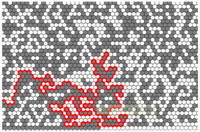
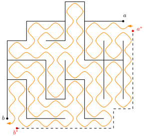
$$\mathcal{G}_{\beta}(x_1, \ldots, x_{2N}) = \prod_{s=1}^{N} |x_{b_s} - x_{a_s}|^{-1/8} \left(\sum_{\sigma \in \{\pm 1\}^N} \prod_{1 \le s < t \le N} \left| \frac{(x_{a_t} - x_{a_s})(x_{b_s} - x_{b_t})}{(x_{b_t} - x_{a_s})(x_{b_s} - x_{a_t})} \right|^{\sigma_s \sigma_t / 4} \right)^{1/2}; \quad (2)$$

the interfaces converge weakly to the Loewner chain associated to $\mathcal{G}_{\beta}$:

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa(\partial_i \log \mathcal{G}_{\beta})(V_t^1, \ldots, V_t^{i-1}, W_t, V_t^{i+1}, \ldots, V_t^{2N})\,\mathrm{d}t, \quad W_0 = x_i, \\ \mathrm{d}V_t^j = \frac{2\,\mathrm{d}t}{V_t^j - W_t}, \quad V_0^j = x_j, \quad j \in \{1, \ldots, i-1, i+1, \ldots, 2N\}, \end{cases} \quad (3)$$

earlier work : [Izyurov, AOP2022] proved (3) for the unnested pattern $\beta = \underline{\cap\cap}$.

Conformal Invariance in 2D Lattice Model

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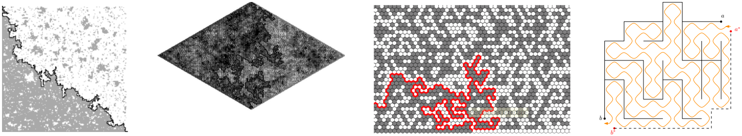

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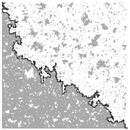
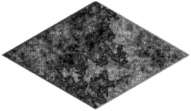
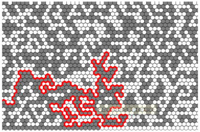
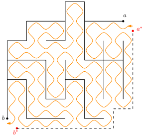

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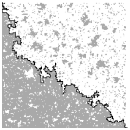
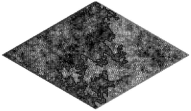
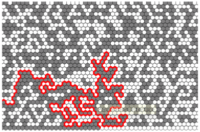
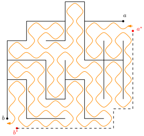

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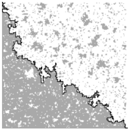
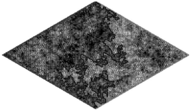
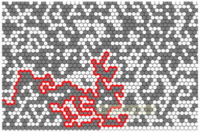
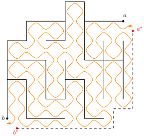

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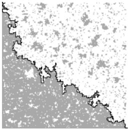
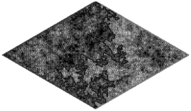
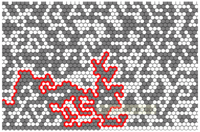
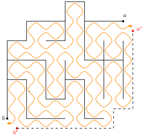

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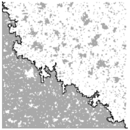
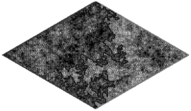
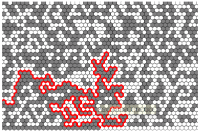
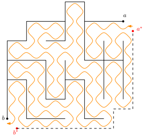
Connection Probabilities

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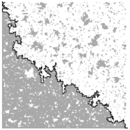 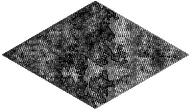 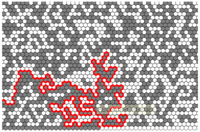 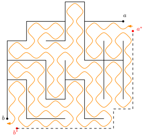

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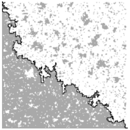
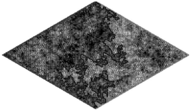
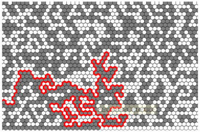
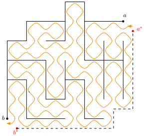

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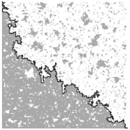
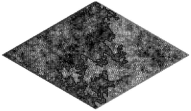
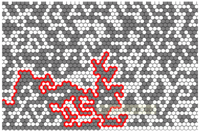
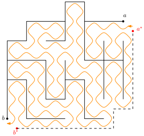

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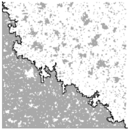

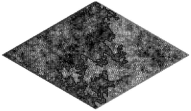

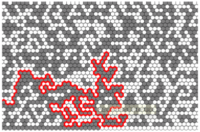

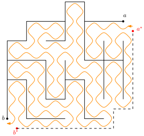

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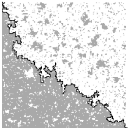

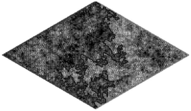

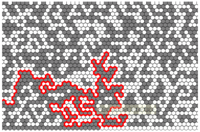

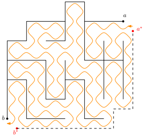

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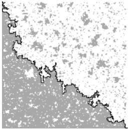
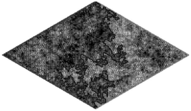
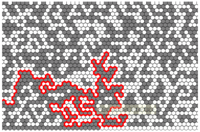
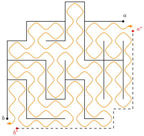

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Strategy :

1. Proper holomorphic observable ϕ_β.
2. Interfaces $\sim$ Loewner chain associated to $\mathcal{G}_\beta$.
3. Analysis on the martingale $\mathcal{Z}_\alpha / \mathcal{G}_\beta$.

Open questions

Conjecture on critical random-cluster model [Flores-Kleban-Simmons-Ziff, 2011, 2017]

Fix parameters $\kappa \in (4, 8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0, 4)$. We have the scaling limit of the connection probabilities :

$$\lim_{\delta \to 0} \mathbb{P}_\beta^\delta[\mathcal{A}^\delta = \alpha] = \mathcal{M}_{\alpha,\beta}(q) \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{G}_\beta(\Omega; x_1, \ldots, x_{2N})}.$$

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Random-cluster model with $q \in [1,4)$ and $\kappa \in (4,6]$.

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- Connection probabilities of the model is known for $q = 2$:
  - A proof without using the convergence of interfaces ? E.g. double-dimer model [Kenyon-Wilson 2011]
  - Generalize it to other $q \in [1, 4)$?

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Random-cluster model with $q \in [1,4)$ and $\kappa \in (4,6]$.

- Convergence of interfaces of the model is known for $q = 2$:
  - Not Known : other $q \in [1,4)$?
- Connection probabilities of the model is known for $q = 2$:
  - A proof without using the convergence of interfaces ? E.g. double-dimer model [Kenyon-Wilson 2011]
  - Generalize it to other $q \in [1,4)$?

Random-cluster model with $q \in (0,1)$ and $\kappa \in (6,8)$.

- The model : Little is known.

Open questions

Conjecture on critical random-cluster model [Flores-Kleban-Simmons-Ziff, 2011, 2017]

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- Coulomb gas integrals $\mathcal{G}_{\beta}$ are well-defined for $\kappa \in (4, 8)$:
  - Not known : positivity for $\kappa \in (6, 8)$? $\mathcal{G}_{\beta} = \sum_{\alpha} \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_{\alpha}$ or a direct proof (e.g. when $\kappa = 8$) ?

Thanks !

1. [Peltola-W. CMP2019] Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Comm. Math. Phys.* 366(2) : 469-536, 2019.
2. [W. CMP2020] Hypergeometric SLE : conformal Markov characterization and applications *Comm. Math. Phys.* 374(2) : 433-484, 2020.
3. [Beffara-Peltola-W. AOP2021] On the uniqueness of global multiple SLEs *Ann. Probab.* 49(1) : 400-434, 2021.
4. [Liu-W. EJP2021] Scaling limits of crossing probabilities in metric graph GFF *Electron. J. Probab.* 26 : article no. 37, 1-46, 2021.
5. [Ding-Wirth-W. AIHP2022] Crossing estimates from metric graph and discrete GFF *Ann. Inst. H. Poincaré Probab. Statist.* 58(3) :1740-1774, 2022.
6. [Peltola-W. AAP2022+] Crossing probabilities of multiple Ising interfaces *Ann. Appl. Probab.* to appear. 2022+
7. [Liu-Peltola-W. 2021] Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithm CFT. arXiv :2108.04421. 2021.
8. [Feng-Peltola-W. 2022] Connection probabilities of multiple FK-Ising interfaces. arXiv :2205.08800. 2022.